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THE BERGMAN PROJECTION OF L^∞ IN TUBES OVER CONES OF REAL, SYMMETRIC, POSITIVE-DEFINITE MATRICES

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ABSTRACT. We determine a defining kernel for the Bergman projection of L^∞ in tubes over cones of real, symmetric, positive-definite matrices.

Let D be a symmetric Siegel domain of type II; let v denote Lebesgue measure in D and $H(D)$ the space of holomorphic functions in D. The Bergman space $A^1(D)$ is defined by $A^1(D) = H(D) \cap L^1(dv)$.

In [**3**], we proved the Coifman-Rochberg conjecture (cf. [**4**]) about the extension to any such domain D of a well-known characterization of the dual of A^1 in the upper half-plane. More precisely, we defined Bloch spaces of holomorphic functions in D and proved that the dual of $A^1(D)$ coincides with each of these spaces. Furthermore, let $\mathcal{B}$ denote a Bloch space in D; we defined in [**3**] the Bergman projection P of L^∞ into $\mathcal{B}$ and proved that P is bounded and onto; consequently, we obtained that the dual of $A^1(D)$ can be realized as the Bergman projection PL^∞ of L^∞.

On the other hand, in the particular cases of the Cayley transform of the unit ball of $\mathbf{C}^n$, $n > 1$, and of the tube over the spherical cone of $\mathbf{R}^{n+1}$, $n \geq 1$, respectively studied in [**1** and **2**], we actually determined a defining kernel for the Bergman projection P of L^∞; the purpose of this paper is to determine an analogous kernel for the tube D in $\mathbf{C}^n$, $n = l(l+1)/2$, over the cone V of $l \times l$ real, symmetric, positive-definite matrices, $l \geq 2$.

In this new domain, we also obtain a kernel satisfying the required property by subtracting from the Bergman kernel $B(\varsigma, z)$ of D a kernel $B_0(\varsigma, z)$ possessing the following properties:

1° the function $\varsigma \mapsto B_0(\varsigma, z)$, $z \in D$ fixed, is holomorphic in D and belongs to the zero equivalence class of the Bloch space $\mathcal{B}$;

2° with respect to z, $B_0(\varsigma, z)$ satisfies the estimate $(B - B_0)(\varsigma, z) \in L^1(dv(z))$, $\varsigma \in D$.

In the first section, we state some preliminary results about the cone V of $l \times l$ real, symmetric, positive-definite matrices.

In the second section, we examine the particular case $l = 2$, where D is the tube in $\mathbf{C}^3$ over the spherical cone of R^3, studied in [**2**]. We shall analyze the construction of the kernel B_0 in that particular case with a view to extending it to the general case; we next resume the proof of the estimate

$$(B - B_0)(\varsigma, z) \in L^1(dv(z)), \qquad \varsigma \in D,$$

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in such a way that an induction idea becomes apparent for the generalization of this estimate to the cone V of rank l: according to this idea, this property should be drawn from previous estimates for the tubes over lower-ranked cones of real, symmetric, positive-definite matrices.

We actually rely on these ideas for the solution of the general case (§3).

According to the classification of symmetric Siegel domains of type II, due to E. Cartan (cf. [**6**, pp. 263–271]), in the following, we study class II (in L. K. Hua's standard notation, also used in [**6**]); on the other hand, we dealt in [**2**] with class IV of tubes over spherical cones of $\mathbf{R}^n$, $n \geq 5$, while the Cayley transform of the ball, studied in [**1**], belongs to class I.

Now, in order to extend this result to any symmetric Siegel domain of type II, it seems likely that our methods can also be successfully applied to classes I and III and the exceptional domains of respective dimensions 16 and 27.

Finally, we resume the notations and terminology introduced in [**3**] and as usual, the same letter "C" will denote constants which may be different from each other.

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1. The cone of $l \times l$ real, symmetric, positive-definite matrices. Let V denote the cone of $l \times l$ real, symmetric, positive-definite matrices

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1l} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2l} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{1l} & \lambda_{2l} & \lambda_{3l} & \cdots & \lambda_{ll} \end{bmatrix}.$$

In the Euclidean space $\mathbf{R}^n$, $n = l(l+1)/2$, represented by the space of $l \times l$ symmetric matrices, the cone V is convex and contains no straight line.

In the following, we use S. G. Gindikin's terminology and notations as introduced in §I.1 of [**3**]. Let us state some properties of the cone V.

The cone V is affine-homogeneous; more precisely, the group $T^+(V)$ of upper triangular matrices with strictly positive diagonal entries acts transitively on V. To prove this claim, it is easy to check that the transformations $\lambda \mapsto g\lambda g'$, where g is a matrix of $T^+(V)$ and g' denotes the transpose matrix of g, are automorphisms of V; the transitivity of the group $T^+(V)$ on V can then be deduced from the fact that any matrix $\lambda \in V$ can be written uniquely in the form $\lambda = tt'$, $t \in T^+(V)$.

We also claim that the cone V has rank l and its given form is canonical. The proof of this claim is made up of the three following facts:

1° The subspace $\mathbf{R}_{ij}^{n_{ij}}$ consists of the $l \times l$ symmetric matrices λ whose entries are all zero except λ_{ij} and λ_{ji}: the common dimension of these subspaces is one and consequently, the cone V is self-conjugate.

2° The cone $V^{(j)}$, $j = 1, \ldots, l$, of the canonical decomposition of the cone V consists of the $l \times l$ symmetric matrices whose $j \times j$ upper principal corner is positive-definite, while all entries outside the $j \times j$ upper principal corner are zero.

3° The forms $F^{(j)}$ are defined as follows. For any λ in V, let C_{j-1} denote the $(j-1) \times (j-1)$ upper principal corner of the matrix $F^{(j)}(\lambda, \lambda)$; the entries of C_{j-1} are $C_{\alpha\beta} = \lambda_{\alpha j}\lambda_{\beta j}$, $1 \leq \alpha \leq \beta \leq j-1$, while all entries in $F^{(j)}(\lambda, \lambda)$ outside C_{j-1} are zero.

The defining functions $\chi_j(\lambda)$ of the cone V are given by

$$\chi_j(\lambda) = \Delta_{l-j+1}(\lambda)/\Delta_{l-j}(\lambda), \qquad j = 1, \dots, l,$$

where $\Delta_i(\lambda)$, $i = 1, \dots, l$, is the $i \times i$ lower principal minor of λ and $\Delta_0(\lambda) = 1$: this is equivalent to the Sylvester criterion for a quadratic form to be positive-definite.

On the other hand, the functions $\chi_j^*(\lambda)$ of V are given by

$$\chi_j^*(\lambda) = \Delta_j^*(\lambda)/\Delta_{j-1}^*(\lambda),$$

where for $i = 1, \dots, l$, $\Delta_i^*(\lambda)$ is the $i \times i$ upper principal minor of λ and $\Delta_0^*(\lambda) = 1$.

Finally, V is self-conjugate with respect to the scalar product $\langle \lambda, \mu \rangle$ = trace of $\lambda\mu'$.

In the particular case $l = 2$, V is the spherical cone Γ of $\mathbf{R}^3$ defined by the inequalities $\chi_j(\lambda) > 0$, $j = 1, 2$, where

$$\chi_2(\lambda) = \lambda_{22}, \quad \chi_1(\lambda) = \lambda_{11} - (\lambda_{12})^2/\lambda_{22},$$

and the functions $\chi_j^*(\lambda)$ are given by

$$\chi_1^*(\lambda) = \lambda_{11}, \quad \chi_2^*(\lambda) = \lambda_{22} - (\lambda_{12})^2/\lambda_{11}.$$

Let us also mention the case $l = 3$. The defining functions $\chi_j(\lambda)$ of the cone V of 3×3 real, symmetric, positive-definite matrices are given by

$$\chi_3(\lambda) = \lambda_{33}, \quad \chi_2(\lambda) = \lambda_{22} - (\lambda_{23})^2/\lambda_{33},$$
$$\chi_1(\lambda) = \lambda_{11} - \frac{(\lambda_{13})^2}{\lambda_{33}} - \frac{(\lambda_{12} - \lambda_{13}\lambda_{23}/\lambda_{33})^2}{\lambda_{22} - (\lambda_{23})^2/\lambda_{33}}$$

and the functions $\chi_j^*(\lambda)$ of V are defined by

$$\chi_1^*(\lambda) = \lambda_{11}, \quad \chi_2^*(\lambda) = \lambda_{22} - (\lambda_{12})^2/\lambda_{11},$$
$$\chi_3^*(\lambda) = \lambda_{33} - \frac{(\lambda_{13})^2}{\lambda_{11}} - \frac{(\lambda_{23} - \lambda_{12}\lambda_{13}/\lambda_{11})^2}{\lambda_{22} - (\lambda_{12})^2/\lambda_{11}}.$$

Moreover, it follows from S. G. Gindikin's results in [**5**] (also cf. §I.1 of [**3**]) that any affine-homogeneous, self-conjugate cone of rank three in $\mathbf{R}^6$ is affine-isomorphic to this cone V.

2. The case $l = 2$. In the case $l = 2$, the domain is the tube $\Omega = \mathbf{R}^3 + i\Gamma$ over the spherical cone Γ of $\mathbf{R}^3$, defined by the inequalities $\chi_j > 0$, $j = 1, 2$, where

$$\chi_2(\lambda) = \lambda_{22}, \quad \chi_1(\lambda) = \lambda_{11} - (\lambda_{12})^2/\lambda_{22};$$

here, $\lambda_{11}, \lambda_{12}$ and λ_{22} denote the coordinates of the point λ of $\mathbf{R}^3 = \mathbf{R}^1_{11} \times \mathbf{R}^1_{12} \times \mathbf{R}^1_{22}$.

The Bergman kernel $B(\varsigma, z)$ of Ω is given by $B(\varsigma, z) = C(\chi_1\chi_2)^{-3}(\varsigma - \bar{z})$ and the Riemann-Liouville differential operator of Ω is the wave operator $\Box$ (box) defined by

$$\Box_\varsigma = 4\frac{\partial^2}{\partial\varsigma_{11}\partial\varsigma_{22}} - \frac{\partial^2}{\partial\varsigma_{12}^2}.$$

We have determined in [**2**] a defining kernel for the Bergman projection of L^∞ in Ω; this kernel is obtained by subtracting from the Bergman kernel B of Ω the kernel B_0 given by

$$(1) \qquad (B - B_0)(\varsigma, z) = C[\chi_1^{-3}(\varsigma - \bar{z}) - \chi_1^{-3}(\tilde{\varsigma} - \bar{z})] \cdot [\chi_2^{-3}(\varsigma - \bar{z}) - \chi_2^{-1/2}(\varsigma - \bar{z})\chi_2^{-5/2}(e - \bar{z})],$$

where in $\mathbf{C}^3 = \mathbf{C}^1_{11} \times \mathbf{C}^1_{12} \times \mathbf{C}^1_{22}$, e is the point of Ω whose coordinates are $e_{11} = e_{22} = i$, $e_{12} = 0$ and for $\varsigma = (\varsigma_{11}, \varsigma_{12}, \varsigma_{22}) \in \Omega$, the point $\tilde{\varsigma}$ of Ω is defined by $\tilde{\varsigma} = (i, 0, \varsigma_{22})$.

Let us shortly analyze the construction of the kernel B_0. We first recall that in Ω,

(i) the Bergman kernel $B(\varsigma, z)$ does not belong to $L^1(dv(z))$, $\varsigma \in \Omega$, because of its bad behaviour when z tends to infinity;

(ii) by Lemma I.2.3 of [**3**], the kernel $B^{1+\alpha}(\varsigma, z)$, $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R}^2$, belongs to $L^1(dv(z))$ for any ς in D if and only if $\alpha_1 > 0$ and $\alpha_2 > 1/6$.

The function $\varsigma \mapsto B_0(\varsigma, z)$, $z \in \Omega$ fixed, must be holomorphic with a zero box in Ω; now, when we look for those couples (β_1, β_2) of $\mathbf{R}^2$ that satisfy the differential equation

$$(\mathrm{E}) \qquad \Box_\varsigma(\chi_1^{-\beta_1}\chi_2^{-\beta_2})(\varsigma - \bar{z}) \equiv 0,$$

we get two kinds of solutions:

(I) $(\beta_1, \frac{1}{2})$, where β_1 is any real number;

(II) $(0, \beta_2)$, where β_2 is any real number.

But, when one refers to §I.2 of [**3**], the vector m is here equal to $(m_1, m_2) = (0, 1)$ and a couple (β_1, β_2) is a solution to equation (E) if and only if one of the β_j's is equal to $m_j/2$, $j = 1, 2$: notice that $\beta_j = m_j/2$ is one of the critical exponents for $(\chi_1^{-\beta_1}\chi_2^{-\beta_2})(\varsigma - \bar{z})$ to be expressed in the integral form given by Lemma I.2.2 of [**3**].

Moreover, since the kernel $B_0(\varsigma, z)$, considered as a function of z, must regularize at once the factors $\chi_1^{-3}(\varsigma - \bar{z})$ and $\chi_2^{-3}(\varsigma - \bar{z})$ of the Bergman kernel $B(\varsigma, z)$, we are led to try the kernel B_0 given in (1); it follows from the next result, proved in [**2**], that this kernel actually satisfies the required properties:

PROPOSITION 2.1. *The kernel B_0 defined in Ω by* (1) *possesses the following properties:*

1° *with respect to ς, $B_0(\varsigma, z)$ is a holomorphic function with a zero box;*

2° *with respect to z, $B_0(\varsigma, z)$ satisfies the estimate $(B - B_0)(\varsigma, z) \in L^1(dv(z))$, $\varsigma \in \Omega$.*

PROOF OF PROPERTY 2°. We modify the proof of property 2° of this proposition, given in [**2**], in such a way that an induction idea becomes apparent for the generalization of this estimate to the general case (l arbitrary).

First, we get from (1) that

$$(2) \qquad |(B - B_0)(\varsigma, z)| = C \prod_{j=1,2} |B_j(\varsigma, z)|,$$

where

$$B_1(\varsigma, z) = \chi_1^{-3}(\varsigma - \bar{z}) - \chi_1^{-3}(\tilde{\varsigma} - \bar{z})$$

and

$$B_2(\varsigma, z) = \chi_1^{-1/2}(\varsigma - \bar{z})[\chi_2^{-5/2}(\varsigma - \bar{z}) - \chi_2^{-5/2}(e - \bar{z})].$$

On the one hand, we use the following easily proved estimate for B_2:

LEMMA 2.1. *For any ς in Ω, there exists a constant $C(\varsigma)$ such that for any z in Ω, the following holds:*

$$|B_2(\varsigma, z)| \leq C(\varsigma)|\chi_2(e - \bar{z})|^{-4}. \tag{3}$$

Now, when one considers the space $\mathbf{R}^3$ in its canonical form $\mathbf{R}^3 = \mathbf{R}^1_{11} \times \mathbf{R}^1_{12} \times \mathbf{R}^1_{22}$, the projection of the cone Γ on the plane $\mathbf{R}^1_{22}$ is contained in the cone $(0, \infty)$ defined in $\mathbf{R}^1_{22}$ by the inequality $\chi_2 > 0$: estimate (3) is therefore supported by the upper half-plane of $\mathbf{C}^1_{22}$.

On the other hand, we shall also use the following estimate for the kernel B_1:

LEMMA 2.2. *For any ς in Ω, there exists a constant $C(\varsigma)$ such that for any z in Ω, the following holds*:

$$|B_1(\varsigma, z)| \leq C(\varsigma)|\chi_1 e(-\bar{z})|^{-4}\left[1 + \left|\frac{\partial}{\partial\varsigma_{12}}\chi_1(e - \bar{z})\right|\right]. \tag{4}$$

PROOF OF LEMMA 2.2. It follows from inequality (3) and from Lemma I.2.5 of [**3**] that for any ς in Ω, there exist two constants $C_1(\varsigma)$ and $C_2(\varsigma)$ such that for any z in Ω, the following inequalities hold:

$$C_1(\varsigma)|\chi_1(e - \bar{z})| \leq |\chi_1(\varsigma - \bar{z})| \leq C_2(\varsigma)|\chi_1(e - \bar{z})|$$

and

$$C_1(\varsigma)\left|\frac{\partial}{\partial\varsigma_{12}}\chi_1(e - \bar{z})\right| \leq \left|\frac{\partial}{\partial\varsigma_{12}}\chi_1(\varsigma - \bar{z})\right| \leq C_2(\varsigma)\left|\frac{\partial}{\partial\varsigma_{12}}\chi_1(e - \bar{z})\right|.$$

Now, Taylor's formula immediately yields the conclusion of the lemma.

Next, to get property 2° of Proposition 2.1, it is enough to prove that the product $K(\varsigma, z)$ of the right-hand sides of inequalities (3) and (4), given by

$$K(\varsigma, z) = C(\varsigma)\sum_{j=1,2}|N_j(z)|,$$

where

$$N_1(z) = (\chi_1\chi_2)^{-4}(e - \bar{z}) \quad \text{and} \quad N_2(z) = \left[(\chi_1\chi_2)^{-4}\frac{\partial}{\partial\varsigma_{12}}\chi_1\right](e - \bar{z}),$$

belongs to $L^1(dv(z))$.

Concerning N_1, the conclusion follows from Lemma I.2.3 of [**3**], because

$$N_1(z) = B^{1+\alpha}(e, z), \qquad \alpha = (1/3, 1/3) > (0, 1/6).$$

For the proof of the property $N_2 \in L^1$, cf. [**2**]. However, as we already noticed in the proof of Lemma 2.2, it appears that to prove that the kernel $(B - B_0)(\varsigma, z)$ defined in the domain $\Omega \subset \mathbf{C}^1_{11} \times \mathbf{C}^1_{12} \times \mathbf{C}^1_{22}$ associated with the two-ranked cone Γ belongs to $L^1(dv(z))$, the first stage is estimate (3) supported by the upper half-plane of C^1_{22} (i.e. the tube over the one-ranked cone of the canonical decomposition of the cone Γ).

Thus, in the general case where the cone V has rank l, to get the analogue of Proposition 2.1, we are naturally led to an induction argument using the analogues of Lemma 2.1 for tubes over the cones $V^{(j)}$, $1 \leq j \leq l - 1$, of the canonical decomposition of the cone V.

3. The general case. Let D denote the tube in $\mathbf{C}^n$, $n = l(l+1)/2$, over the cone V of $l \times l$ real, symmetric, positive-definite matrices. The defining functions χ_j and χ_j^*, $j = 1, \ldots, l$, of the cone V are explicitly given in the first paragraph and the canonical form of the space $\mathbf{C}^n$ containing the domain D is $\mathbf{C}^n = \prod_{1 \le \sigma \le \tau \le l} \mathbf{C}^1_{\sigma\tau}$.

In the following, we shall suppose $l \geq 3$ and we use the terminology and notations introduced in the first two chapters of [**3**]. In particular, $\mathcal{R}$ here denotes the differential polynomial in $\mathbf{C}^n$ that possesses the property

$$\mathcal{R}_\varsigma \exp(\langle \lambda, \varsigma \rangle) = (\lambda^*)^\rho \exp(\langle \lambda, \varsigma \rangle), \tag{5}$$

where the vector ρ of $\mathbf{N}^l$ is $\rho = (l, l-1, \ldots, 1)$.

Moreover, we shall denote by $\mathcal{B}$ the Bloch space $\mathcal{B}_{\rho,r}$ corresponding to this vector ρ and to the vector r of $\mathbf{N}^l$ whose coordinates r_j are given by

$$r_j = j2^{j^2} + \sum_{\sigma=j+1}^{l} r_\sigma, \qquad r_l = l2^{l^2}, \tag{6}$$

and $\mathcal{D}$ will denote the Riemann-Liouville differential operator of D defined by $\mathcal{D} = \mathcal{R}^r$.

Let us now begin the construction of the kernel B_0 of D. We first recall that in D,

(i) the Bergman kernel $B(\varsigma, z)$ given by

$$B(\varsigma, z) = c(\varsigma - \bar{z})^{2d}, \tag{7}$$

where d is the vector of $\mathbf{R}^l$ whose coordinates are all equal to $-(l+1)/2$, does not belong to $L^1(dv(z))$, $\varsigma \in D$, because of its bad behaviour when z tends to infinity;

(ii) by Lemma I.2.3 of [**3**], the kernel $B^{1+\alpha}(\varsigma, z)$, $\alpha \in \mathbf{R}^l$, belongs to $L^1(dv(z))$ for any ς in D if and only if $\alpha > m/2(l+1)$, where m is the vector of N^l whose coordinates are $m_j = j - 1$, $j = 1, \ldots, l$.

Secondly, like in the case $l = 2$, the kernel $B_0(\varsigma, z)$ must satisfy the two following properties:

(iii) with respect to ς, $B_0(\varsigma, z)$ is a holomorphic function satisfying $\mathcal{D}_\varsigma B_0(\varsigma, z) \equiv 0$;

(iv) with respect to z, $B_0(\varsigma, z)$ must regularize at one and the same time each of the factors $\chi_1^{-(l+1)}(\varsigma - \bar{z}), \ldots, \chi_l^{-(l+1)}(\varsigma - \bar{z})$ of the expression (7) of the Bergman kernel $B(\varsigma, z)$ of D, when z tends to infinity, so that $(B - B_0)(\varsigma, z)$ belongs to $L^1(dv(z))$ for any ς in D.

Following the same lines as in the case $l = 2$, let us first consider the differential equation

$$\mathcal{D}_\varsigma(\chi_1^{-\beta_1} \cdots \chi_l^{-\beta_l})(\varsigma - \bar{z}) \equiv 0, \tag{E}$$

where $\beta = (\beta_1, \ldots, \beta_l)$ is a vector of $\mathbf{R}^l$. We claim that any vector β of $\mathbf{R}^l$ satisfying $\beta \geq m/2$, with at least one coordinate β_j equal to $m_j/2$ (β is one of the critical multi-indices for $(z)^{-\beta}$ to be expressed in the integral form given in Lemma I.2.2 of [**3**]) gives rise to a solution of (E); this is proved in the following more general form:

PROPOSITION 3.1. *Let the index j take the values $2,\dots,l$ and let β_i, $i = 1,\dots,j-1$, denote real numbers satisfying $\beta_i > m_i/2$.*

1° *Assume first that $j \le l-1$; for any C^∞ function $\varphi_{j+1}(\varsigma)$ of the only coordinates $\varsigma_{\sigma\tau}$, $\sigma \ge j+1$, of ς, the following holds:*

$$\mathcal{R}_\varsigma(\chi_1^{-\beta_1}\cdots\chi_{j-1}^{-\beta_{j-1}}\chi_j^{-m_j/2}\varphi_{j+1})(\varsigma) \equiv 0. \tag{8}$$

2° *For $j = l$, one has the equality*

$$\mathcal{R}_\varsigma(\chi_1^{-\beta_1}\cdots\chi_{l-1}^{-\beta_{l-1}}\chi_l^{-m_l/2})(\varsigma) \equiv 0. \tag{9}$$

We recall that $\mathcal{R}$ is the differential polynomial in $\mathbf{C}^n$ that possesses property (5) and $\mathcal{D}$ is given by $\mathcal{D} = \mathcal{R}^r$, where r is the vector of $\mathbf{N}^l$ defined in (6), $r > (1,\dots,1)$.

PROOF. We first prove property 2°. According to the notations of the first section, $\mathcal{R}$ is the differential polynomial in $\mathbf{C}^n$ that possesses the property

$$\begin{aligned}\mathcal{R}_\varsigma \exp(\langle\lambda,\varsigma\rangle) &= (\Delta_1^*\Delta_2^*\cdots\Delta_l^*)(\lambda)\exp(\langle\lambda,\varsigma\rangle)\\ &= (\chi_1^{*l}\chi_2^{*l-1}\cdots\chi_l^*)(\lambda)\exp(\langle\lambda,\varsigma\rangle).\end{aligned} \tag{5}$$

Now, if Λ_l denotes the differential polynomial in $\mathbf{C}^n$ that possesses the property

$$\begin{aligned}(\Lambda_l)_\varsigma \exp(\langle\lambda,\varsigma\rangle) &= \Delta_l^*(\lambda)\exp(\langle\lambda,\varsigma\rangle)\\ &= (\chi_1^*\chi_2^*\cdots\chi_l^*)(\lambda)\exp(\langle\lambda,\varsigma\rangle),\end{aligned} \tag{10}$$

property (9) will immediately follow from the identity

$$(\Lambda_l)_\varsigma(\chi_1^{-\beta_1}\cdots\chi_{l-1}^{-\beta_{l+1}}\chi_l^{-m_l/2})(\varsigma) \equiv 0. \tag{11}$$

Let us prove (11). We introduce a complex variable β_l and we denote by Π the half-plane $\Pi = \{\beta_l \in \mathbf{C}\colon \operatorname{Re}\beta_l > m_l/2\}$.

By Lemma I.2.2 of [**3**], we have for any β_l in Π:

$$\left(\prod_{j=1}^{l-1}\chi_j^{-\beta_j}\chi_l^{-\beta_l}\right)(\varsigma) = C_\beta\int_V \exp(i\langle\lambda,\varsigma\rangle)\left(\prod_{j=1}^{l-1}\chi_j^{*\beta_j-(l+1)/2}\chi_l^{*\beta_l-(l+1)/2}\right)(\lambda)\,d\lambda;$$

here, S. G. Gindikin [**5**] actually determined that $C_\beta = C_{\beta_1\cdots\beta_{l-1}}/\Gamma(\beta_l - m_l/2)$.

Next, in view of (10), we obtain that for any β_l in Π

$$\begin{aligned}(\Lambda_l)_\varsigma\left(\prod_{j=1}^{l-1}\chi_j^{-\beta_j}\chi_l^{-\beta_l}\right)(\varsigma) &= C_{\beta_1\cdots\beta_{l-1}}/\Gamma(\beta_l - m_l/2)\\ &\cdot\int_V \exp(i\langle\lambda,\varsigma\rangle)\left(\prod_{j=1}^{l-1}\chi_j^{*\beta_j+1-(l+1)/2}\chi_l^{*\beta_l+1-(l+1)/2}\right)(\lambda)\,d\lambda,\end{aligned}$$

and it follows from Lemma I.2.2 of [**3**] that

$$\begin{aligned}(\Lambda_l)_\varsigma\left(\prod_{j=1}^{l-1}\chi_j^{-\beta_j}\chi_l^{-\beta_l}\right)(\varsigma) &= C'_{\beta_1\cdots\beta_{l-1}}\Gamma(\beta_l+1-m_l/2)/\Gamma(\beta_l - m_l/2)\\ &\cdot\left(\prod_{j=1}^{l-1}\chi_j^{-\beta_j-1}\chi_l^{-\beta_l-1}\right)(\varsigma).\end{aligned} \tag{12}$$

We then point out that, when ς belongs to D, the left-hand side of (12) defines an entire function of β_l while the right-hand side defines a holomorphic function of β_l in the half-plane Π. On the other hand, when β_l tends to $m_l/2$ from inside Π, the right-hand side of (12) tends to zero because the quotient $\Gamma(\beta_l+1-m_l/2)/\Gamma(\beta_l-m_l/2)$ tends to zero; therefore, the left-hand side of (12), whose limit is precisely the value at $\beta_l = m_l/2$, also tends to zero: this proves (11) and consequently (9).

We next prove property 1° of the proposition. Equality (8) will be obtained in the same way as (9) by giving respectively the preceding roles of $\beta_l = m_l/2$ and of the differential polynomial Λ_l to $\beta_j = m_j/2$ and to the differential polynomial Λ_j in $\mathbf{C}^n$ that possesses the property

$$(\Lambda_j)_\varsigma \exp(\langle\lambda,\varsigma\rangle) = \Delta_j^*(\lambda)\exp(\langle\lambda,\varsigma\rangle) = (\chi_1^*\chi_2^*\cdots\chi_j^*)(\lambda)\exp(\langle\lambda,\varsigma\rangle). \tag{13}$$

More precisely, to obtain (7), it is enough to show that

$$(\Lambda_1)_\varsigma \varphi_2(\varsigma) \equiv 0 \tag{14}$$

and that for $j = 2,\ldots,l-1$,

$$(\Lambda_j)_\varsigma \left(\prod_{\sigma=1}^{j-1} \chi_\sigma^{-\beta_\sigma}\chi_j^{-m_j/2}\varphi_{j+1}\right)(\varsigma) \equiv 0. \tag{15}$$

The proof of equality (14) is straightforward, because $(\Lambda_1)_\varsigma = \partial/\partial\varsigma_{11}$ and the function $\varphi_2(\varsigma)$ does not depend on ς_{11}.

Concerning equality (15), since the differential operator (Λ_j) does not act on the coordinates $\varsigma_{\sigma\tau}$, $\sigma \geq j+1$, it suffices to prove that $(\Lambda_j)_\varsigma(\prod_{\sigma=1}^{j-1}\chi_\sigma^{-\beta_\sigma}\chi_j^{-m_j/2})(\varsigma) \equiv 0$; furthermore, when one notices that the function $\prod_{k=j+1}^{l}\chi_k^{-(l+1)}(\varsigma)$ only depends on the coordinates $\varsigma_{\sigma\tau}$, $\sigma \geq j+1$, of ς, the conclusion will follow from the equality

$$(\Lambda_j)_\varsigma \left(\prod_{\sigma=1}^{j-1}\chi_\sigma^{-\beta_\sigma}\chi_j^{-m_j/2}\prod_{k=j+1}^{l}\chi_k^{-(l+1)}\right)(\varsigma) \equiv 0:$$

now, the proof of this last equality is similar to that of (11). This entirely proves Proposition 3.1.

We next give the expression of the kernel B_0 that will be subtracted from the Bergman kernel B of D; B_0 is defined by

$$(B-B_0)(\varsigma,z) = [B(\varsigma,z) - B(\varsigma^1,z)]\prod_{j=2}^{l} Q^{(j)}(\varsigma,z), \tag{16}$$

where the point ς^1 of D and the kernels $Q^{(j)}$, $j = 2,\ldots,l$, are defined as follows.

We denote by $\varsigma_{\sigma\tau}$ the coordinates of the point ς of D with respect to the canonical decomposition of $\mathbf{C}^n$, $\mathbf{C}^n = \prod_{1\leq\sigma\leq\tau\leq l}\mathbf{C}^1_{\sigma\tau}$, the point ς^1 is then given by its coordinates $\varsigma^1_{11} = i$, $\varsigma^1_{1\tau} = 0$ for $\tau = 2,\ldots,l$, $\varsigma^1_{\sigma\tau} = \varsigma_{\sigma\tau}$ for $1 < \sigma \leq \tau \leq l$.

The kernels $Q^{(j)}$, $j = 2,\ldots,l$, are given by

$$Q^{(j)}(\varsigma,z) = 1 - \chi_j^{-(l+R_j-m_j/2)}(\varsigma^j - \bar{z}) \cdot \sum_{\substack{k_j\in\mathbf{N}^{l-j+1}\\ |k_j|\leq R_j-1}} \frac{(\varsigma^j-\varsigma)^{k_j}}{k_j!}\left(\frac{\partial}{\partial\varsigma}\right)^{k_j}\chi_j^{l+R_j-m_j/2}(\varsigma-\bar{z}), \tag{17}$$

where

(I) $R = (R_1, \dots, R_l)$ is the vector of $\mathbf{N}^l$ whose coordinates are $R_j = j2^{j^2}$; notice that the coordinates of the vector r of $\mathbf{N}^l$ defined in (6) (in view of the definition of the Riemann-Liouville differential operator $\mathcal{D}$ of D as $\mathcal{D} = \mathcal{R}^r$) are given by $r_j = \sum_{\sigma=j}^{l} R_\sigma$;

(II) ς^j denotes the point of D whose coordinates are

$$\begin{aligned} \varsigma^j_{\sigma\sigma} &= i && \text{for } \sigma = 1, \dots, j, \\ \varsigma^j_{\sigma\tau} &= 0 && \text{for } \sigma < \tau,\ \sigma = 1, \dots, j, \text{ and} \\ \varsigma^j_{\sigma\tau} &= \varsigma_{\sigma\tau} && \text{for } j+1 \le \sigma \le \tau \le l; \end{aligned}$$

(III) k_j is a vector of $\mathbf{N}^{l-j+1}$ whose coordinates are denoted by $k_{j\tau}$, $\tau = j, \dots, l$; we set

$$|k_j| = \sum_{\tau=j}^{l} k_{j\tau}, \quad k_j! = \prod_{\tau=j}^{l} k_{j\tau}!$$

and by the symbols $(\varsigma^j - \varsigma)^{k_j}$ and $(\partial/\partial\varsigma)^{k_j}$, we mean

$$(\varsigma^j - \varsigma)^{k_j} = \prod_{\tau=j}^{l} (\varsigma^j - \varsigma)_{j\tau}^{k_{j\tau}} \quad \text{and} \quad \left(\frac{\partial}{\partial\varsigma}\right)^{k_j} = \prod_{\tau=j}^{l} \left(\frac{\partial}{\partial\varsigma_{j\tau}}\right)^{k_{j\tau}}.$$

We next prove the following proposition:

PROPOSITION 3.2. *For any z in D, the kernel $B_0(\varsigma, z)$ satisfies $\mathcal{D}_\varsigma B_0(\varsigma, z) \equiv 0$.*

PROOF. It is enough to prove the two following lemmas:

LEMMA 3.1. *Λ_l denotes the differential polynomial defined in $\mathbf{C}^n$ by (10). For any z in D, the following holds:*

$$(\Lambda_l^{R_l})_\varsigma \{B(\varsigma, z)[1 - Q^{(l)}(\varsigma, z)]\} \equiv 0.$$

LEMMA 3.2. *Let the index j take one of the values $1, \dots, l-1$ and let Λ_j denote the corresponding differential polynomial defined in $\mathbf{C}^n$ by (13).*

For any z in D and any function $\varphi_{j+1}(\varsigma, z)$, $\mathcal{C}^\infty$ with respect to ς and only depending on the coordinates $\varsigma_{\sigma\tau}$, $\sigma \ge j+1$ of ς, the following identities hold:

1° $(\Lambda_1)_\varsigma \{B(\varsigma^1, z)\varphi_2(\varsigma, z)\} \equiv 0$;

2° $(\Lambda_j^{R_j})_\varsigma \{B(\varsigma, z)[1 - Q^{(j)}(\varsigma, z)]\varphi_{j+1}(\varsigma, z)\} \equiv 0$ *if* $2 \le j \le l-1$.

PROOF OF LEMMA 3.1. There exist a function $\varphi(z)$ and constants C_k, $k = 0, 1, \dots, R_{l-1}$ such that

$$1 - Q^{(l)}(\varsigma, z) = \varphi(z) \sum_{k=0}^{R_l - 1} C_k (i - \varsigma_{ll})^k \cdot \chi_l^{(l+1)/2 + R_l - k}(\varsigma - \bar{z}).$$

It then suffices to show that for any $k = 0, 1, \dots, R_l - 1$, the following identity holds:

$$(18) \qquad (\Lambda_l^{R_l})_\varsigma \left\{ \left(\prod_{j=1}^{l-1} \chi_j^{-(l+1)} \chi_l^{-(l+1)/2 + R_l - k} \right) (\varsigma - \bar{z})(i - \varsigma_{ll})^k \right\} \equiv 0.$$

Let us first prove (18) for $k = R_l - 1$, i.e.

$$(\Lambda_l^{R_l})_\varsigma\{F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l - 1}\} \equiv 0,$$

where

$$F_l(\varsigma, z) = \left(\prod_{j=1}^{l-1} \chi_j^{-(l+1)} \chi_l^{-(l-1)/2}\right)(\varsigma - \bar{z}) \tag{19}$$

is a kernel which, by (11), satisfies $(\Lambda_l)F_l(\varsigma, z) \equiv 0$.

We recall that Λ_l is the differential polynomial associated in $\mathbf{C}^n$ with the polynomial $\Lambda_l^*(\lambda)$; similarly, Λ_{l-1} is the differential polynomial associated with the polynomial $\Lambda_{l-1}^*(\lambda)$ which is considered here as the minor of $\Lambda_l^*(\lambda)$ corresponding to the entry λ_{ll}. Thus, since $(\Lambda_l)_\varsigma F_l(\varsigma, z) \equiv 0$, we obtain

$$\begin{aligned}(\Lambda_l)_\varsigma\{F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-1}\} &= C_l(\Lambda_{l-1})_\varsigma F_l(\varsigma, z)\frac{\partial}{\partial \varsigma_{ll}}(i - \varsigma_{ll})^{R_l-1} \\ &= C_l'(\Lambda_{l-1})_\varsigma F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-2}, \\ (\Lambda_l^{R_l-1})_\varsigma\{F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-1}\} &= C_l(\Lambda_{l-1}^{R_l-1})_\varsigma F_l(\varsigma, z)\end{aligned}$$

and it then follows that

$$(\Lambda_l^{R_l})_\varsigma\{F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-1}\} \equiv 0;$$

this proves (18) for $k - R_l - 1$.

Let us next prove equality (18) for $k = R_l - 2$, i.e. $(\Lambda_l^{R_l})_\varsigma G(\varsigma, z) \equiv 0$, where

$$G(\varsigma, z) = F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-2}\chi_l(\varsigma - \bar{z});$$

here, F_l again denotes the kernel defined in (19).

Now, since $\chi_l(\varsigma - \bar{z}) = (\varsigma - \bar{z})_{ll} = (i - \bar{z}_{ll}) - (i - \varsigma_{ll})$, it is enough to prove that the kernels

$$G_1(\varsigma, z) = F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-1} \quad \text{and} \quad G_2(\varsigma, z) = F_l(\varsigma, z)(i - \varsigma_{ll})^{R_l-2}$$

satisfy $(\Lambda_l^{R_l})_\varsigma G_j(\varsigma, z) \equiv 0,\ j = 1, 2$.

The proof for G_1 amounts to the preceding case; concerning G_2, the conclusion follows from the same argument as in the case $k = R_l - 1$.

In the same way, one successively proves equality (18) for $k = R_l - 3, \ldots, 1$ and 0. The proof of Lemma 3.1 is complete.

PROOF OF LEMMA 3.2. The proof of identity 1° of this lemma is straightforward since $(\Lambda_1)_\varsigma = \partial/\partial\varsigma_{11}$ and the function $B(\varsigma^1, z)\varphi_2(\varsigma, z)$ does not depend on the coordinate ς_{11} of ς.

Let us next prove identity 2°, first in the case $j = 2$. Since by definition,

$$\begin{aligned}1 - Q^{(2)}(\varsigma, z) &= \chi_2^{-(l+R_2-1/2)}(\varsigma^2 - \bar{z}) \\ &\quad \cdot \sum_{\substack{k_2 \in \mathbf{N}^{l-1} \\ |k_2| \le R_2 - 1}} \frac{(\varsigma^2 - \varsigma)^{k_2}}{k_2!}\left(\frac{\partial}{\partial\varsigma}\right)^{k_2} \chi_2^{l+R_2-1/2}(\varsigma - \bar{z}),\end{aligned}$$

it suffices to show that for any vector $k = k_2$ of $\mathbf{N}^{l-1}$, $|k| \le R_2 - 1$, if we set

$$G_k(\varsigma, z) = \left[\prod_{j=1}^{l} \chi_j^{-(l+1)}\left(\frac{\partial}{\partial\varsigma}\right)^k \chi_2^{l+R_2-1/2}\right](\varsigma - \bar{z})(\varsigma^2 - \varsigma)^k\varphi_3(\varsigma, z),$$

where the function $\varphi_3(\varsigma, z)$ only depends on the coordinates $\varsigma_{\sigma\varsigma}$, $\sigma \geq 3$, we have

$$(\Lambda_2^{R_2})_\varsigma G_k(\varsigma, z) \equiv 0. \tag{20}$$

Let us recall that in (13), Λ_2 is defined by

$$(\Lambda_2)_\varsigma = 4\frac{\partial^2}{\partial\varsigma_{11}\partial\varsigma_{22}} - \frac{\partial^2}{\partial\varsigma_{12}^2};$$

we then point out that the factors $(\varsigma^2 - \varsigma)_{2\tau}^{k_{2\tau}}$, $\tau \geq 3$, of $(\varsigma^2 - \varsigma)^k$, where $k = k_2 = (k_{22}, k_{23}, \ldots, k_{2l})$, and the functions do not depend on $\varsigma_{11}, \varsigma_{12}$ and ς_{22}, while it is easy to check that

$$\left(\frac{\partial}{\partial\varsigma}\right)^k \chi_2^{l+R_2-1/2}(\varsigma - \bar{z}) = \chi_2^{l+R_2-1/2-|k|}(\varsigma - \bar{z})\varphi(\varsigma, z),$$

where $\varphi(\varsigma, z)$ is a function which does not depend on the coordinates ς_{11}, ς_{12} and ς_{22} of ς.

Now, to get (20), it is enough to prove that

$$(\Lambda_2^{R_2})_\varsigma\{F_2(\varsigma, z)\chi_2^{R_2-1-|k|}(\varsigma - \bar{z})(i - \varsigma_{22})^{k_{22}}\} \equiv 0,$$

where $F_2(\varsigma, z) = (\chi_1^{-(l+1)}\chi_2^{-1/2})(\varsigma - \bar{z})$.

Notice that by (15), F_2 satisfies $(\Lambda_2)_\varsigma F_2(\varsigma, z) \equiv 0$; identity (20) is then obtained like (18): more precisely, the role played in the proof of (18) by the kernel F_l defined in (19) is here given to the kernel F_2.

The case $j = 2$ in the lemma is thus solved and the cases $j = 3, \ldots, l-1$ can be treated in the same way: details are omitted. This proves Lemma 3.2, and consequently the proof of Proposition 3.2 is complete.

Our next concern is the following proposition:

PROPOSITION 3.3. *The kernel B_0 defined in* (16) *satisfies the estimate $(B - B_0)(\varsigma, z) \in L^1(dv(z))$, for any ς in D.*

PROOF. We first prove the following lemma:

LEMMA 3.3. *Let ε_j, $j = 2, \ldots, l$, denote strictly positive real numbers and e the point of $D \subset \prod_{1\leq\sigma\leq\tau\leq l} \mathbf{C}^1_{\sigma\tau}$ whose coordinates are $e_{\sigma\sigma} = 1$ for $\sigma = 1, \ldots, l$ and $e_{\sigma\tau} = 0$ for $1 \leq \sigma < \tau \leq l$.*

The function F_k, $k = 1, \ldots, l$, defined in D by

$$F_k(z) = \left(\frac{\partial}{\partial\varsigma_{1k}}\chi_1^{-(l+1)}\prod_{j=2}^{l}\chi_j^{-(l+1+m_j/2+\varepsilon_j)}\right)(e - \bar{z})$$

belongs to $L^1(D)$.

PROOF OF LEMMA 3.3. In the case $k = 1$,

$$F_1(z) = C(\chi_1^{-(l+2)}\prod_{j=2}^{l}\chi_j^{-(l+1+m_j/2+\varepsilon_j)})(e - \bar{z})$$

is in the form $B^{1+\alpha}(e, z)$, $\alpha > m/2(l+1)$: the conclusion then follows from Lemma I.2.3 of [**3**].

We next consider the case $k = 2$. Let α_j, $j = 1, 2, \ldots, l$, denote real numbers satisfying $0 < \alpha_1 < 1$ and $m_j/2 < \alpha_j < 2\varepsilon_j + m_j/2$ if $j \geq 2$; we divide the domain D into $D = D_1 \cup D_2$, where

$$D_1 = \left\{ z \in D \colon |F_2(z)| \leq \left| \prod_{j=1}^{l} \chi_j^{-(l+1+\alpha_j)}(e - \bar{z}) \right| \right\}, \qquad D_2 = D \backslash D_1.$$

First observe that by Lemma I.2.3 of [**3**], the integral of $|F_2(z)|$ over D_1 converges because

$$|F_2(z)| \leq |B^{1+\alpha/(l+1)}(e, z)|, \qquad \alpha > m/2,$$

for any z in D_1.

It is then enough to prove that the integral of $|F_2(z)|$ over D_2 also converges. Since for any z in D_2, the following inequality holds:

$$|F_2(z)| \leq |F_2(z)|^2 \left| \prod_{j=1}^{l} \chi_j^{l+1+\alpha_j}(e - \bar{z}) \right| = |G(z)|^2,$$

where

$$G(z) = \frac{\partial}{\partial \varsigma_{12}} \left(\chi_1^{-(l+1-\alpha_1)/2} \prod_{j=2}^{l} \chi_j^{-[(l+j-\alpha_j)/2+\varepsilon_j]} \right)(e - \bar{z}),$$

it suffices to show that the function G belongs to $L^2(D)$. We use the fact that by Lemma I.2.2 of [**3**], G can be written in the form

$$G(z) = C \int_V \lambda_{12} \left(\chi_1^{*-\alpha_1/2} \prod_{j=2}^{l} \chi_j^{*(j-\alpha_j-1)/2+\varepsilon_j} \right)(\lambda) \cdot \exp(i\langle \lambda, e - \bar{z} \rangle)\, d\lambda.$$

Lemma I.2.1 of [**3**] and the following lemma then yield the desired conclusion:

LEMMA 3.4. *For any real numbers α_j, $j = 1, \ldots, l$, satisfying $\alpha_1 < 1$ and $\alpha_j < 2\varepsilon_j + m_j/2$ if $j \geq 2$ the integral*

$$I_l = \int_V (\lambda_{12})^2 \left(\chi_1^{*-\alpha_1-(l+1)/2} \prod_{j=2}^{l} \chi_j^{*j-\alpha_j+2\varepsilon_j-(l+3)/2} \right)(\lambda) \exp\left(-2 \sum_{j=1}^{l} \lambda_{jj} \right) d\lambda$$

is convergent.

PROOF OF LEMMA 3.4. The proof is by induction on $l \geq 2$. For $l = 2$, we have

$$I_2 = \int_\Gamma (\lambda_{12})^2 (\chi_1^{*-\alpha_1-3/2} \chi_2^{*-1/2-\alpha_2+2\varepsilon_2})(\lambda) \exp[-2(\lambda_{11} + \lambda_{22})]\, d\lambda_{11}\, d\lambda_{12}\, d\lambda_{22},$$

where Γ is the spherical cone of $\mathbf{R}^3$ and $\varepsilon_2 > 0$. It then follows from calculations in [**2**] that the integral I_2 converges when $\alpha_1 < 1$ and $\alpha_2 < 2\varepsilon_2 + 1/2$.

Assume now that when the cone V has rank $l - 1$, the integral I_{l-1} converges when $\alpha_1 < 1$ and $\alpha_j < 2\varepsilon_j + m_j/2$ for $j = 2, \ldots, l-1$. We then deduce from this assumption that the integral I_l over the l-ranked cone V converges when $\alpha_1 < 1$ and $\alpha_j < 2\varepsilon_j + m_j/2$ for $j = 2, \ldots, l$.

First integrate I_l with respect to λ_{ll}, using the change of variable $t = \chi_l^*(\lambda)$; since $\alpha_l < 2\varepsilon_l + (l-1)/2$, we obtain

$$I_l = C\int (\lambda_{12})^2 \left(\chi_1^{*-\alpha_1-(l+1)/2} \prod_{j=2}^{l-1} \chi_j^{*j-\alpha_j+2\varepsilon_j-(l+3)/2}\right)(\lambda) \cdot \exp\left(-2\prod_{j=1}^{l-1}\lambda_{jj}\right) \exp[-2(\lambda_{ll} - \chi_l^*(\lambda)] \prod_{(j,k)\neq(l,l)} d\lambda_{jk}.$$

Next integrate successively with respect to $\lambda_{l-1,l}, \lambda_{l-2,l}, \ldots, \lambda_{1l}$, since

$$\int_{\mathbf{R}^{l-1}} \exp[-2(\lambda_{ll} - \chi_l^*(\lambda))]\, d\lambda_{l-1,l}\cdots d\lambda_{1l} = C(\chi_1^*\cdots\chi_{l-1}^*)^{1/2}(\lambda),$$

we get the equality $I_l = C_l I_{l-1}$ when $\alpha_l < 2\varepsilon_l + (l-1)/2$ and the induction assumption yields the desired conclusion: the proof of Lemma 3.4 is complete.

We have just proved Lemma 3.3 for $k = 1, 2$; the conclusion for $k = 3, \ldots, l$ can be obtained in the same way: this entirely proves Lemma 3.3.

Now, to get the conclusion of Proposition 3.3, it suffices, by Lemma 3.3, to prove the following lemma:

LEMMA 3.5. *For any ς in D, there exist a constant $C(\varsigma)$ and strictly positive real numbers ε_j, $j = 2, \ldots, l$, such that for any z in D, the following inequality holds*:

$$(21) \qquad |(B - B_0)(\varsigma, z)| \leq C(\varsigma)\left(1 + \sum_{k=2}^{l}\left|\frac{\partial}{\partial\varsigma_{1k}}\chi_1(e-\bar{z})\right|\right) \cdot \left|\left(\chi_1^{-(l+2)}\prod_{j=2}^{l}\chi_j^{-(l+1+m_j/2+\varepsilon_j)}\right)(e-\bar{z})\right|.$$

To prove Lemma 3.5, we use the following lemma:

LEMMA 3.6. *Let $Q^{(j)}$, $j = 2, \ldots, l$, denote the kernels defined in* (17). *For any ς in D there exists a constant $C(\varsigma)$ such that for any $j = 2, \ldots, l$ and z in D the following inequality holds*:

$$(22) \qquad |Q^{(j)}(\varsigma, z)| \leq C(\varsigma)|\chi_j^{-(l+R_j-m_j/2)}(e-\bar{z})| \cdot \sum_{\substack{k_j\in\mathbf{N}^{l-j+1}\\ |k_j|=R_j}} \left|\left(\frac{\partial}{\partial\varsigma}\right)^{k_j}\chi_j^{l+R_j-m_j/2}(e-\bar{z})\right|.$$

PROOF OF LEMMA 3.6. For $j = l$, one has

$$Q^{(l)}(\varsigma, z) = 1 - \chi_l^{-[(l+1)/2+R_l]}(\varsigma^l - \bar{z}) \cdot \sum_{k=0}^{R_l-1} C_k(i - \varsigma_{ll})^k \chi_l^{(l+1)/2+R_l-k}(\varsigma - \bar{z});$$

in this case, estimate (22) is supported by the upper half-plane of $\mathbf{C}_{ll}^1$ and follows immediately from Taylor's formula and from the following easily proved estimate:

for any w in D, there exist two constants $C_1(w)$ and $C_2(w)$ such that for any z in D, one has

$$C_1(w)|\chi_l(e-\bar{z})| \leq |\chi_l(w-\bar{z})| \leq C_2(w)|\chi_l(e-\bar{z})|. \tag{23}$$

For $j = l-1$, estimate (22) is supported by the tube Ω in $\prod_{l-1\leq\sigma<\tau\leq l} \mathbf{C}^1_{\sigma\tau}$ over the spherical cone $V^{(2)}$ of $\mathbf{R}^3$, studied in the preceding section. By (23) and using Lemma I.2.5 of [**3**] in the case of the domain Ω, we get that for any w in D, there exist two constants $C_1(w)$ and $C_2(w)$ such that for any z in D, we have

$$C_1(w)|\chi_{l-1}(e-\bar{z})| \leq |\chi_{l-1}(w-\bar{z})| \leq C_2(w)|\chi_{l-1}(e-\bar{z})| \tag{24}$$

and

$$\begin{aligned} C_1(w) \sum_{\substack{k\in\mathbf{N}^2\\ |k|=R_{l-1}}} &\left| \left(\frac{\partial}{\partial\varsigma}\right)^k \chi_{l-1}^{l/2+R_{l-1}+1}(e-\bar{z}) \right| \\ \leq \sum_{\substack{k\in\mathbf{N}^2\\ |k|=R_{l-1}}} &\left| \left(\frac{\partial}{\partial\varsigma}\right)^k \chi_{l-1}^{l/2+R_{l-1}+1}(w-\bar{z}) \right| \\ \leq C_2(w) \sum_{\substack{k\in\mathbf{N}^2\\ |k|=R_{l-1}}} &\left| \left(\frac{\partial}{\partial\varsigma}\right)^k \chi_{l-1}^{l/2+R_{l-1}+1}(e-\bar{z}) \right|; \end{aligned} \tag{25}$$

estimate (22) for $j = l-1$ then follows from Taylor's formula and inequalities (24) and (25).

In the same way, we conclude successively for the cases $j = l-2, l-3, \ldots, 2$, using Lemma I.2.5 of [**3**] and inequalities of the same type as (23), (24) and (25), supported by the tubes over the cones $V^{(3)}, \ldots, V^{(j)}$ of the canonical decomposition of the cone V. Lemma 3.6 is then entirely proved.

We also use the following lemma:

LEMMA 3.7. *Let ς be a point of D and ς^1 the point of D defined in* (16). *There exists a constant $C(\varsigma)$ such that for any z in D, the following holds*:

$$\begin{aligned} &|B(\varsigma, z) - B(\varsigma^1, z)| \\ &\quad\leq C(\varsigma)\left(1 + \sum_{k=2}^{l} \left|\left(\frac{\partial}{\partial\varsigma_{1k}}\right)\chi_1(e-\bar{z})\right|\right)\left|\chi_1^{-(l+2)} \prod_{j=2}^{l} \chi_j^{-(l+1)}(e-\bar{z})\right|. \end{aligned}$$

PROOF OF LEMMA 3.7. By Lemma I.2.5 of [**3**] and using inequalities of the same type as (23) and (24), supported by the tubes over the cones $V^{(j)}$, $j = 2, \ldots, l$, we obtain that for any w in D, there exist two constants $C_1(w)$ and $C_2(w)$ such that for any z in D, the following hold:

$$C_1(w)|\chi_1(e-\bar{z})| \leq |\chi_1(w-\bar{z})| \leq C_2(w)|\chi_1(e-\bar{z})| \tag{26}$$

and

$$\begin{aligned} C_1(w)\sum_{k=1}^{l}\left|\frac{\partial}{\partial w_{1k}}\chi_1(e-\bar{z})\right| &\leq \sum_{k=1}^{l}\left|\frac{\partial}{\partial w_{1k}}\chi_1(w-\bar{z})\right| \\ &\leq C_2(w)\sum_{k=1}^{l}\left|\frac{\partial}{\partial w_{1k}}\chi_1(e-\bar{z})\right|. \end{aligned} \tag{27}$$

Taylor's formula, inequalities (26) and (27) and the fact that $(\partial/\partial\varsigma_{11})\chi_1(\varsigma-\bar{z})\equiv 1$ then yield the conclusion of Lemma 3.7.

PROOF OF LEMMA 3.5. In view of Lemmas 3.6 and 3.7, it is enough to prove that there exist strictly positive real numbers ε_j, $j=2,\dots,l$, such that for any j, if k_j denotes a vector of $\mathbf{N}^{l-j+1}$ satisfying $|k_j|=R_j$, the function $F(z)=\prod_{j=2}^{l}F_j(z)$, where

$$F_j(z)=\left[\chi_j^{-(l-m_j+R_j-\varepsilon_j)}\left(\frac{\partial}{\partial\varsigma}\right)^{k_j}\chi_j^{l-m_j/2+R_j}\right](e-\bar{z}) \tag{28}$$

is bounded in D.

The proof is easy when all the k_j's are in the form $k_j=(R_j,0,\dots,0)$; in this case, $F_j(z)=C_j\chi_j^{-(R_j-m_j/2-\varepsilon_j)}(e-\bar{z})$. Since for any $j=2,\dots,l$ and any z in D, one has $|\chi_j(e-\bar{z})|\geq 1$ and $R_j-m_j/2>0$, the conclusion follows by taking $\varepsilon_j<R_j-m_j/2$.

Actually, among all the differential operators $\partial/\partial\varsigma_{j\tau}$, $\tau=j,\dots,l$, the most regularizing is $\partial/\partial\varsigma_{jj}$, while the other operators $\partial/\partial\varsigma_{j\tau}$, $\tau\neq j$, have the same regularity. Hence, for the other values of the k_j's, we only give the proof in the case when $k_j=(0,R_j,0,\dots,0)$, for any $j=2,\dots,l-1$ and $k_l=R_l$; this is done by successively estimating the functions $F_l, F_{l-1}F_l,\dots,F_2\cdots F_l$.

Concerning F_l, one has

$$F_l(z)=C_l\chi_l^{-[R_l-(l-1)/2-\varepsilon_l]}(e-\bar{z}); \tag{29}$$

taking ε_l such that $0<\varepsilon_l<R_l-(l-1)/2$, we find that the function F_l is bounded in D because, for any z in D, $|\chi_l(e-\bar{z})|\geq 1$.

Let us next consider the product $F_{l-1}F_l$. We are going to prove the following lemma:

LEMMA 3.8. *For any strictly positive real numbers $\varepsilon_{l-1},\varepsilon_l,\eta_{l-1}>\varepsilon_{l-1}$ and $n_l>\varepsilon_l$, one has*

$$\begin{aligned}(30)\quad &|(F_{l-1}F_l)(z)|\\ &\leq C_l|(\chi_{l-1}^{-[R_{l-1}/2-(l-2)/2-\eta_{l-1}]}\chi_l^{-[R_l-R_{l-1}/4-(l-1)/2-\eta_l]})(e-\bar{z})|.\end{aligned}$$

PROOF OF LEMMA 3.8. By (28), for $k_{l-1}=(0,R_{l-1})$, we get

$$(F_{l-1}F_l)(z)=\chi_{l-1}^{-(R_{l-1}+2-\varepsilon_{l-1})}\left(\frac{\partial}{\partial\varsigma_{l-1,l}}\right)^{R_{l-1}}G(z),$$

where

$$G(z)=\left(\chi_{l-1}^{l/2+R_{l-1}+1}\chi_l^{-[R_l-(l-1)/2-\varepsilon_l]}\right)(e-\bar{z}).$$

Let $\Box$ denote the wave operator defined by

$$\Box_\varsigma=4\frac{\partial^2}{\partial\varsigma_{l-1,l-1}\partial\varsigma_{ll}}-\frac{\partial^2}{\partial\varsigma_{l-1,l}^2};$$

it is easy to check that for any real numbers α and β, one has

$$\Box(\chi_{l-1}^\alpha\chi_l^\beta)=C_{\alpha,\beta}\chi_{l-1}^{\alpha-1}\chi_l^{\beta-1}.$$

Now, since

$$\frac{\partial^2}{\partial \varsigma_{l-1,l}^2} = -\Box_\varsigma + 4\frac{\partial^2}{\partial \varsigma_{l-1,l-1}\partial \varsigma_{ll}},$$

we obtain

$$(31)\quad (F_{l-1}F_l)(z) = \chi_{l-1}^{-(R_{l-1}+2-\varepsilon_{l-1})} \sum_{p=0}^{R_{l-1}/2} C_p \Box_\varsigma^{R_{l-1}/2-p} \left(\frac{\partial^2}{\partial \varsigma_{l-1,l-1}\partial \varsigma_{ll}}\right)^p G(z),$$

where G again denotes the function defined at the beginning of the proof.

Hence, as soon as we point out that, in the right side of (31), the differential operator $\Box_\varsigma$ is more regularizing than $\partial^2/\partial\varsigma_{l-1,l-1}\partial\varsigma_{ll}$, we find it sufficient to prove that for any strictly positive real numbers $\varepsilon_{l-1}, \varepsilon_l, \eta_{l-1} > \varepsilon_{l-1}$ and $\eta_l > \varepsilon_l$, the modulus of the last term (corresponding to $p = R_{l-1}/2$) in the right side of (31) is inferior to the right side of (30).

Since in that term, we have

$$\begin{aligned} C_l &\left(\frac{\partial}{\partial \varsigma_{ll}}\right)^{R_{l-1}/2} \left(\chi_{l-1}^{(l+R_{l-1})/2+1} \chi_l^{-[R_l-(l-1)/2-\varepsilon_l]}\right)(e-\bar z) \\ &= \sum_{q=0}^{R_{l-1}/2} C_q \left[\left(\frac{\partial}{\partial \varsigma_{ll}}\right)^q \chi_{l-1}^{(l+R_{l-1})/2+1} \left(\frac{\partial}{\partial \varsigma_{ll}}\right)^{R_{l-1}/2-q} \chi_l^{-[R_l-(l-1)/2-\varepsilon_l]}\right](e-\bar z) \\ &= \sum_{q=0}^{R_{l-1}/2} C_q' \chi_{l-1}^{(l+R_{l-1})/2-q+1} \chi_l^{-[R_l+(R_{l-1}-l+1)/2-q-\varepsilon_l]} \left(\frac{\partial}{\partial \varsigma_{ll}}\chi_{l-1}\right)^q (e-\bar z), \end{aligned}$$

the conclusion follows from the following lemma:

LEMMA 3.9. *For any strictly positive real numbers δ_{l-1} and δ_l, the function*

$$\varphi(z) = \left[\chi_{l-1}^{-(1+\delta_{l-1})} \left(\frac{\partial}{\partial \varsigma_{ll}}\chi_{l-1}\right) \chi_l^{-(1/2+\delta_l)}\right](e-\bar z)$$

is bounded in D.

PROOF OF LEMMA 3.9. It is easy to check that $\varphi = C_1\varphi_1 + C_2\varphi_2$, where

$$\begin{aligned} \varphi_1(z) &= \left(\chi_{l-1}^{-\delta_{l-1}} \frac{\partial}{\partial \varsigma_{ll}} \chi_l^{-(1/2+\delta_l)}\right)(e-\bar z) \\ &= C_l \left(\chi_{l-1}^{-\delta_{l-1}} \chi_l^{-(3/2+\delta_l)}\right)(e-\bar z) \end{aligned}$$

and

$$\varphi_2(z) = \frac{\partial}{\partial \varsigma_{ll}} \left(\chi_{l-1}^{-\delta_{l-1}} \chi_l^{-(1/2+\delta_l)}\right)(e-\bar z).$$

Since the function φ_1 is bounded in D, we just have to prove that φ_2 is also bounded in D. Applying Lemma I.2.2 of [**3**] in the tube over the spherical cone $\Gamma = V^{(2)}$ of $\mathbf{R}^3 = \prod_{l-1\le\sigma\le\tau\le l} \mathbf{R}^1_{\sigma\tau}$ yields

$$|\varphi_2(z)| \le C_l \int_\Gamma \lambda_{ll} \left(\overset{*}{\chi}{}_{l-1}^{\delta_{l-1}-3/2} \overset{*}{\chi}{}_l^{\delta_l-1}\right)(\lambda) \cdot \exp(-\lambda_{l-1,l-1} - \lambda_{ll}) \prod_{l-1\le\sigma\le\tau\le l} d\lambda_{\sigma\tau};$$

the conclusion then follows from calculations made in [**2**]: this proves Lemma 3.9 and hence, the proof of Lemma 3.8 is complete.

PROOF OF LEMMA 3.5 (CONTINUED). It suffices to prove that for any strictly positive real numbers ε_σ and $\eta_\sigma > \varepsilon_\sigma$, $\sigma = j, \ldots, l$, one has

$$\prod_{\sigma=j}^{l} |F_\sigma(z)| \le C_j \left| \left[\chi_j^{-[(R_j-j+1)/2-\eta_j]} \cdot \prod_{\sigma=j+1}^{l-1} \chi_\sigma^{-[R_\sigma/2-\sum_{p=1}^{\sigma-j}(pR_{\sigma-p})/4-(\sigma-1)/2-\eta_\sigma]} \cdot \chi_l^{-[R_l-\sum_{p=1}^{l-j}(pR_{l-p})/4-(l-1)/2-\eta_j]} \right] (e-\bar{z}) \right|. \tag{32}$$

First notice that in the extreme case $j=2$, the exponents $(R_2-1)/2$, $R_\sigma/2 - \sum_{p=1}^{\sigma-2}(pR_{\sigma-p})/4-(\sigma-1)/2$ for $3 \le \sigma \le l-1$ and $R_l - \sum_{p=1}^{l-2}(pR_{l-p})/4-(l-1)/2$ are all strictly positive.

The proof of (32) is by induction on j. By Lemma 3.8, estimate (32) holds for $j = l-1$. Assume now that for any strictly positive real numbers ε_σ and $\eta_\sigma > \varepsilon_\sigma$, $\sigma = j+1, \ldots, l$, we have

$$\prod_{\sigma=j+1}^{l} |F_\sigma(z)| \le C_j \left| \left[\chi_{j+1}^{-[(R_{j+1}-j)/2-\eta_{j+1}]} \cdot \prod_{\sigma=j+2}^{l-1} \chi_\sigma^{-[R_\sigma/2-\sum_{p=1}^{\sigma-j-1}(pR_{\sigma-p})/4-(\sigma-1)/2-\eta_\sigma]} \cdot \chi_l^{-[R_l-\sum_{p=1}^{l-j-1}(pR_{l-p})/4-(l-1)/2-\eta_l]} \right] (e-\bar{z}) \right|. \tag{33}$$

Next notice that the left side of (32) is the product of the left side of (33) by the modulus of the function F_j defined in (28) and here, $k_j = (0, R_j, 0, \ldots, 0)$. We then estimate the function F_j in the same way as in the case $j = l-1$ (Lemma 3.8). More precisely, we introduce the wave operator $\square$ defined by

$$\square_\varsigma = 4\frac{\partial^2}{\partial\varsigma_{jj}\partial\varsigma_{j+1,j+1}} - \frac{\partial^2}{\partial\varsigma_{j,j+1}^2}$$

on the tube in $\prod_{j\le\sigma\le\tau\le l} \mathbf{C}^1_{\sigma\tau}$ over the cone $V^{(l-j+1)}$; as in the proof of Lemma 3.8, we here use the induction assumption (33) to prove that the product $\prod_{\sigma=j}^{l} |F_\sigma(z)|$ is inferior to the product of (33) by

$$\left| \left(\chi_j^{-[(R_j-j+1)/2-\delta_j]} \prod_{\sigma=j+1}^{l} \chi_\sigma^{R_j(\sigma-j)/4+\delta_\sigma} \right) (e-\bar{z}) \right|,$$

where δ_σ, $\sigma = j, \ldots, l$, denotes any strictly positive real number. To get this last estimate, we use the following generalization of Lemma 3.9:

LEMMA 3.10. *Let the integer j take the values $2, \ldots, l-1$. For any strictly positive real numbers δ_σ, $\sigma = j, \ldots, l$, the function*

$$\left\{ \chi_j^{-(1+\delta_j)} \left(\frac{\partial}{\partial \varsigma_{j+1,j+1}} \chi_j \right) \prod_{\sigma=j+1}^{l} \chi_\sigma^{-[(\sigma-j)/2+\delta_\sigma]} \right\} (e - \bar{z})$$

is bounded in D.

The proof of Lemma 3.10 is very similar to that of Lemma 3.9 and uses Lemma I.2.3 of [**3**] in the tube over the cone $V^{(l-j+1)}$: details are omitted.

The proof of Lemma 3.5 is now complete and hence, Proposition 3.3 is entirely proved.

We next recall that $\mathcal{B} = \mathcal{B}_{\rho,r}$ denotes the Bloch space of D corresponding to the vectors ρ and r of $\mathbf{N}^l$, where $\rho = (l, l-1, \ldots, 1)$ and r is defined in (6); let us denote by P the Bergman projection defined in [**3**] as a bounded operator from L^∞ onto the Bloch space $\mathcal{B}$ of D.

Propositions 3.2 and 3.3 then lead to the following theorem:

THEOREM 3.1. *Let B_0 denote the kernel defined in D by* (16). *For any bounded function b in D,*

1° *one defines a holomorphic function g in D by*

$$g(\varsigma) = \int_D (B - B_0)(\varsigma, z) b(z)\, dv(z), \qquad \varsigma \in D; \tag{34}$$

2° *the element Pb of the Bloch space $\mathcal{B}$* (*the Bergman projection of b*) *can be represented by the holomorphic function g.*

PROOF. In view of Proposition 3.3, the expression (34) of g actually defines a function. To prove that this function is holomorphic, it suffices to show that

$$\frac{\partial}{\partial \varsigma_{jk}} g(\varsigma) = \int_D \frac{\partial}{\partial \varsigma_{jk}} (B - B_0)(\varsigma, z) b(z)\, dv(z),$$

i.e. the following lemma holds:

LEMMA 3.11. *For every compact K of D and for every ς in K, there exists an integrable function $M(z)$ in D such that for any integers j and k satisfying $1 \le j \le k \le l$, the following holds*:

$$\left| \frac{\partial}{\partial \varsigma_{jk}} (B - B_0)(\varsigma, z) \right| \le M(z).$$

PROOF OF LEMMA 3.11. For $j = k = 1$, we have

$$\frac{\partial}{\partial \varsigma_{11}} (B - B_0)(\varsigma, z) = \left(\chi_1^{-(l+2)} \prod_{j=2}^{l} \chi_j^{-(l+1)} \right) (\varsigma - \bar{z}) \prod_{j=2}^{l} Q^{(j)}(\varsigma, z);$$

the conclusion for this derivative then follows from the proof of Proposition 3.3.

Concerning the other derivatives, the ingredients in the proof of Proposition 3.3 also yield the desired conclusion; furthermore, notice that those derivatives behave better than the terms stemming from $(B - B_0)(\varsigma, z)$. This proves Lemma 3.11, and consequently part 1° of the theorem is proved.

On the other hand, for part 2°, the conclusion easily follows from the equality

$$\mathcal{D}_\varsigma B(\varsigma, z) = cB^{1+r\rho/(l+1)}(\varsigma, z)$$

and from Proposition 3.2: the proof of Theorem 3.1 is then complete.

REMARK. We just determined a defining kernel for the Bergman projection of L^∞ onto a particular Bloch space $\mathcal{B} = \mathcal{B}_{\rho,r}$ in the tube over the cone of $l \times l$ real, symmetric, positive-definite matrices, $l \geq 2$. However, for $l \geq 3$, we also defined in [**3**] the Bergman projection of L^∞ onto a Bloch space corresponding to smaller vectors ρ and r, e.g. $\rho = (1, \ldots, 1)$ and r is the product of ρ by the smallest integer strictly greater than $(l-1)/2$; it would be interesting to determine a defining kernel for the Bergman projection of L^∞ onto the Bloch space corresponding to these values of ρ and r that are indeed the smallest possible.

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